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Exact solutions for the coagulation–fragmentation equation

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Abstract. We present new exact equilibrium and time-dependent solutions to the two main versions of the coagulation–fragmentation equation. Equilibria are first found for the pure coagulation and pure fragmentation equations; equilibrium solutions for the full coagulation–fragmentation equation are then constructed when the kernels governing the reaction rates are chosen to be in a particular form. Time-dependent solutions are also derived for the pure fragmentation equation.

1. Introduction

In this article we study the nonlinear kinetic equation

$$\begin{aligned}\frac{\partial}{\partial t}c(m,t)=\tfrac{1}{2}\int_0^m K(m-m_1,m_1)c(m-m_1,t)c(m_1,t)\,\mathrm{d}m_1\\ -\int_0^\infty K(m,m_1)c(m,t)c(m_1,t)\,\mathrm{d}m_1\\ +\int_m^\infty \gamma(m_1,m)c(m_1,t)\,\mathrm{d}m_1-\frac{c(m,t)}{m}\int_0^m m_1\gamma(m,m_1)\,\mathrm{d}m_1 \qquad (1)\end{aligned}$$

with $t \geqslant 0$ and initial data $c(m,0)=c_0(m)\geqslant 0$ for $m\geqslant 0$. Equation (1) is known as the general coagulation–fragmentation equation; it describes the time evolution of particles $c(m,t)$ of mass $m\geqslant 0$ undergoing a change in mass governed by the non-negative reaction rates $K(m,m_1)$ (the coagulation kernel) and $\gamma(m,m_1)$ (the birth rate of m_1-mass particles due to fragmentation of m-mass particles). The first and third terms on the right-hand side of (1) describe a growth in the number of particles of mass m due to coagulation and fragmentation respectively, while the second and fourth terms describe the reverse of these processes. Coagulation and fragmentation equations arise in a number of problems including reacting polymers, clustering of colloidal particles, astrophysics and birth–death processes [1].

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If it is assumed that each particle can only be split into two sub-particles then $\gamma(m, m_1) = \gamma(m, m - m_1)$ and equation (1) may be rewritten in the form [2-4]

$$\frac{\partial}{\partial t} c(m,t) = \frac{1}{2}\int_0^m K(m-m_1, m_1)c(m-m_1,t)c(m_1,t)\,\mathrm{d}m_1 - \int_0^\infty K(m,m_1)c(m,t)c(m_1,t)\,\mathrm{d}m_1 + \int_0^\infty F(m,m_1)c(m+m_1,t)\,\mathrm{d}m_1 - \frac{1}{2}\int_0^m F(m-m_1,m_1)c(m,t)\,\mathrm{d}m_1 \tag{2}$$

where the fragmentation kernel becomes $F(m-m_1, m_1) = \gamma(m, m_1)$. It is important to note tht the kernels K and F are symmetric non-negative functions and that the fragmentation model in equation (1) is more general than that in equation (2). For a more extensive discussion see [3-5].

The main objective of this paper is to present a number of new exact equilibrium and time-dependent solutions to equations (1) and (2). Other exact solutions may be found in [6, 7] while information on existence, uniqueness, density conservation and other properties may be found in [2, 5, 6, 8-16].

2. Equilibrium solutions for the pure coagulation equation

We shall first consider the time-independent equilibrium equation for (1) when the fragmentation kernel $\gamma \equiv 0$, that is, when

$$\frac{1}{2}\int_0^m K(m-m_1,m_1)c(m-m_1)c(m_1)\,\mathrm{d}m_1 - \int_0^\infty K(m,m_1)c(m)c(m_1)\,\mathrm{d}m_1 = 0. \tag{3}$$

By setting $g(m, m_1) = K(m, m_1)c(m)c(m_1)$ we may rewrite (3) as

$$\frac{1}{2}\int_0^m g(m-m_1,m_1)\,\mathrm{d}m_1 = \int_0^\infty g(m,m_1)\,\mathrm{d}m_1. \tag{4}$$

It is straightforward to check that there can be no continuous non-trivial non-negative solutions to (4); indeed, integrating (4) with respect to m results in the contradiction

$$\frac{1}{2}\int_0^\infty\int_0^\infty g(m,m_1)\,\mathrm{d}m_1\,\mathrm{d}m = \int_0^\infty\int_0^\infty g(m,m_1)\,\mathrm{d}m_1\,\mathrm{d}m. \tag{5}$$

Consequently, any non-negative solution of (4) must possess singularities.

We now try to find a solution to (4) of the following form:

$$g(m,m_1) = g(m_1,m) = \begin{cases} f(am+m_1) & m \leqslant m_1 \\ f(am_1+m) & m \geqslant m_1 \end{cases} \tag{6}$$

where a is a constant. Substituting (6) into (4) (and splitting the integration on the left hand side of (4) into integrations over $[0, m/2]$ and $[m/2, m]$) shows that

$$\int_0^{m/2} f((a-1)m_1+m)\,\mathrm{d}m_1 = \int_0^m f(am_1+m)\,\mathrm{d}m_1 + \int_m^\infty f(am+m_1)\,\mathrm{d}m_1. \tag{7}$$

If it is assumed that $a \neq 0$ and $a \neq 1$ then

$$\frac{1}{(a-1)}\int_m^{(a+1)m/2} f(y)\,\mathrm{d}y = \frac{1}{a}\int_m^{(a+1)m} f(y)\,\mathrm{d}y + \int_{(a+1)m}^\infty f(y)\,\mathrm{d}y. \tag{8}$$

Differentiating (8) with respect to m yields

$$\frac{1}{(a-1)}\left\{\frac{(a+1)}{2}f((a+1)m/2)-f(m)\right\}=\frac{(1-a^2)}{a}f((a+1)m)-\frac{1}{a}f(m). \tag{9}$$

Solutions of the form $f(m)=m^k$ will now be sought. The power k is necessarily less than -1 because one of the integrals in (8) is over an infinite region; we must also therefore have $a>-1$, otherwise the regions of integration in (8) include the singular point $y=0$.

It is easily verified that for any constant f_0

$$f(m)=f_0m^{-3} \tag{10}$$

is the solution of (9) for the above-mentioned functions for all $a>-1$ (by inspection the restrictions on a may be removed and the cases $a=0$ and $a=1$ can be included.) It now follows that if for any non-negative function $v(m)\geqslant 0$

$$K(m, m_1)=\begin{cases} v(m)v(m_1)(am+m_1)^{-3} & m\leqslant m_1 \\ v(m)v(m_1)(am_1+m)^{-3} & m\geqslant m_1 \end{cases} \tag{11}$$

where $a>-1$, then for any constant A

$$c(m)=\frac{A}{v(m)} \tag{12}$$

is an equilibrium solution to (1) with $\gamma\equiv 0$.

3. Equilibrium solutions for the pure fragmentation equation

In this section we shall consider the equilibrium solutions to equations (1) and (2) when the coagulation kernel $K\equiv 0$. For equation (1) we require

$$\int_m^\infty \gamma(m_1, m)c(m_1)\,\mathrm{d}m_1-\frac{c(m)}{m}\int_0^m m_1\gamma(m, m_1)\,\mathrm{d}m_1=0. \tag{13}$$

Upon setting $g(m+am_1)=\gamma(m_1, m)c(m_1)m$ equation (13) may be rewritten as

$$\int_m^\infty g(m+am_1)\,\mathrm{d}m_1=\int_0^m g(m_1+am)\,\mathrm{d}m_1. \tag{14}$$

Differentiating (14) with respect to m gives

$$g(m+am)=a(a+2+1/a)^{-1}g(am) \tag{15}$$

from which it follows that for any constant g_0

$$g(m)=g_0m^{-2}. \tag{16}$$

Hence if for any function $v(m)\geqslant 0$ we have

$$\gamma(m_1, m)=\frac{v(m_1)}{m}(m+am_1)^{-2} \tag{17}$$

then for any constant A

$$c(m)=\frac{A}{v(m)} \tag{18}$$

is a solution of (13). It should be noted that in this case $\gamma(m_1, m) \neq \gamma(m_1, m_1 - m)$ and therefore the solution (18) cannot solve the form of equation given by (2). Nevertheless, the techniques of section 2 above may be employed to derive an equilibrium solution for (2); namely, if

$$F(m, m_1) = \begin{cases} v(m+m_1)(am+m_1)^{-3} & m \leqslant m_1 \\ v(m+m_1)(am_1+m)^{-3} & m \geqslant m_1 \end{cases} \tag{19}$$

where $a > -1$ and $v(m) \geqslant 0$, then for any constant A

$$c(m) = \frac{A}{v(m)} \tag{20}$$

is an equilibrium solution for equation (2) when $K \equiv 0$.

4. Equilibrium solutions to the coagulation–fragmentation equation

We shall now find equilibrium solutions to both forms of the full coagulation-fragmentation equation given in (1) and (2). Our investigation begins with an examination of equation (1).

Let us suppose that in (1) there is a function f which satisfies

$$K(m, m_1)c(m)c(m_1) = mc(m_1)\gamma(m_1, m) = f(m+m_1). \tag{21}$$

We then obtain from (1)

$$\tfrac{1}{2}mf(m) - \int_m^\infty f(y)\,\mathrm{d}y + \frac{1}{m}\int_{2m}^\infty f(y)\,\mathrm{d}y - \frac{1}{m}\int_m^{2m} f(y)\,\mathrm{d}y = 0. \tag{22}$$

Differentiating (22) with respect to m gives

$$\tfrac{1}{2}m^3 f'(m) + (\tfrac{3}{2}m^2 + m)f(m) - 4mf(2m) = \int_{2m}^\infty f(y)\,\mathrm{d}y - \int_m^{2m} f(y)\,\mathrm{d}y. \tag{23}$$

A further differentiation yields the differential equation

$$\frac{m^2}{2}f''(m) + (3m+1)f'(m) - 8f'(2m) + 3f(m) = 0 \tag{24}$$

which has the solution $f(m) = m^{-2}$; but this solution does not satisfy the original equation. We therefore seek a solution of the form

$$f(m) = m^{-3} + a_4 m^{-4} + a_5 m^{-5} + \dots \tag{25}$$

where the a_k are constants. Substituting (25) into (24) results in the recurrence relation

$$a_3 = 1 \qquad a_k = \frac{2(k-1)(1-2^{3-k})}{(k-2)(k-3)}a_{k-1} \qquad k \geqslant 4 \tag{26}$$

from which we find that

$$a_k = 2^{k-4}\frac{(k-1)}{(k-3)!}\prod_{i=4}^{k}(1-2^{3-i}) \qquad k \geqslant 4 \tag{27}$$

and

$$f(m) = \sum_{k=3}^{\infty} a_k m^{-k}. \tag{28}$$

It is simple to verify that the above series converges for any $m>0$. Thus, if for any $v(m)\geqslant 0$

$$K(m, m_1)=v(m)v(m_1)\sum_{k=3}^{\infty} a_k(m+m_1)^{-k} \tag{29}$$

and

$$\gamma(m_1, m)=\frac{1}{m}v(m_1)\sum_{k=3}^{\infty} a_k(m+m_1)^{-k} \tag{30}$$

then

$$c(m)=\frac{1}{v(m)} \tag{31}$$

is an equilibrium solution to equation (1).

We now consider the coagulation-fragmentation equation in the form (2) and proceed to use the results from sections 2 and 3 above. If for $v(m)\geqslant 0$ the kernels K and F are given as in equations (11) and (19) for $a>-1$ then for any scalar λ

$$c(m)=\frac{e^{\lambda m}}{v(m)} \tag{32}$$

is an equilibrium solution to (2); this is clearly seen to be true upon inserting the relation

$$K(m, m_1)c(m)c(m_1)=F(m, m_1)c(m+m_1) \tag{33}$$

into equation (2). In general, if $c(m)$ is an equilibrium solution of (2) which satisfies equation (33) for given kernels K and F then it follows that $e^{\lambda m}c(m)$ is also an equilibrium solution for any constant λ.

If, however, $c(m)$ is any equilibrium solution to equation (2) then a number of kernels K and F which correspond to this solution may be found by choosing the kernels such that they satisfy the relation

$$K(m, m_1)c(m)c(m_1)-F(m, m_1)c(m+m_1)=\begin{cases}(am+m_1)^{-3} & m\leqslant m_1\\ (am_1+m)^{-3} & m\geqslant m_1\end{cases} \tag{34}$$

where $a>-1$. This follows from the arguments in section 2 using equations (4), (6) and (10); the equilibrium equation is then clearly satisfied. As an example, $c(m)=e^{\lambda m}$ is an equilibrium solution for the case when $K\equiv F\equiv 1$, by equation (33). Further, by (34), it is also a solution when

$$K(m, m_1)=(1+(m+m_1)^{-3})\,e^{-\lambda(m+m_1)} \qquad F(m, m_1)=e^{-\lambda(m+m_1)}. \tag{35}$$

5. Time-dependent solutions

In this section we derive time-dependent solutions to (2) in the pure fragmentation case ($K\equiv 0$); exact time-dependent solutions for $F\equiv 0$ have recently been discussed in [6]. Suppose

$$F(m, m_1)=2F_0(m+m_1)^k \qquad k>-1 \tag{36}$$

where F_0 is a constant. Motivated by the $k=0$ result in [6] we make the ansatz

$$c(m, t)=e^{\lambda t}(\lambda+F_0m^p)^{-q} \tag{37}$$

where $\lambda > 0$ and p and q are to be determined. Substituting this expression into equation (2) and multiplying throughout by $e^{-\lambda t}$ yields

$$(\lambda + F_0 m^{k+1})(\lambda + F_0 m^p)^{-q} = 2F_0 \int_m^\infty m_1^k (\lambda + F_0 m_1^p)^{-q} \, dm_1. \tag{38}$$

Differentiation of (38) with respect to m and dividing by F_0 gives (provided $k - pq < -1$)

$$\begin{aligned}(k+1)m^k(\lambda + F_0 m^p)^{-q} - m^{p-1}pq(\lambda + F_0 m^{k+1})(\lambda + F_0 m^p)^{-q-1} \\ = -2m^k(\lambda + F_0 m^p)^{-q}.\end{aligned} \tag{39}$$

Hence, by setting $p = k+1$ and dividing by $m^k(\lambda + F_0 m^{k+1})^{-q}$ we obtain

$$(k+1)(1-q) = -2 \tag{40}$$

from which it is seen that $c(m, t)$ given by equation (37) is a time-dependent solution to (2) provided

$$p = k+1 \qquad q = (k+3)(k+1)^{-1}. \tag{41}$$

We also have $k - pq = -3$ which shows that equations (38) and (39) are valid; moreover, from equations (37) and (41), $1 - pq < -1$ and therefore

$$\int_0^\infty mc(m, t) \, dm < \infty \tag{42}$$

for all finite times. In this case the solutions are not density conserving and it should be noted that condition (36) does not satisfy the uniqueness hypothesis of Stewart [8]. Other solutions to (2) for $K \equiv 0$ and $F = (m + m_1)^k$, $k > -1$, have been found explicitly by Ziff and McGrady [7] for particular values of $k > -1$ when the initial data is exponentially decaying.

A further time-dependent solution to (2) may be constructed when $K \equiv 0$ and

$$c(m, t) = \sum_{j=1}^{\infty} c_j(t)\delta(m - j) \tag{43}$$

where δ is the Dirac delta function. In this case the resulting discrete version of (2) may be written as

$$\frac{d}{dt} c_j(t) = -\tfrac{1}{2} \sum_{i=1}^{j-1} F(j-i, i)c_j(t) + \sum_{i=j+1}^{\infty} F(i-j, j)c_i(t) \tag{44}$$

where $t \geq 0$ and $j = 1, 2, 3, \ldots$. We consider the case

$$F(i, j) = i + j. \tag{45}$$

Define the sequence X_j, $j = 1, 2, 3, \ldots$, by

$$X_1 = 1 \qquad X_{j+1} = X_j(1 + 1/j)^3(1 - (4j+2)(j^2 + 3j + 4)^{-1}). \tag{46}$$

Upon inserting the ansatz

$$c_j(t) = e^t j^{-3} X_j \tag{47}$$

into equation (44) it is clear that (47) is a solution provided

$$j^{-3}(j^2 - j + 2)X_j/2 = \sum_{i=j+1}^{\infty} i^{-2} X_i. \tag{48}$$

From (46) it follows that $X_j \to L$ for some L as $j \to \infty$. Consider (46) rewritten as (with j replaced by i)

$$i^{-3}(i^2-i+2)X_i/2-(i+1)^{-3}((i+1)^2-(i+1)+2)X_{i+1}/2=(i+1)^{-2}X_{i+1}. \tag{49}$$

Since $X_i \to L$ expression (49) can be summed from j to infinity to give

$$j^{-3}(j^2-j+2)X_j/2=\sum_{i=j}^{\infty}(i+1)^{-2}X_{i+1} \tag{50}$$

from which equation (48) is shown to be true; thus (47) is a time-dependent solution to (44). Since the right-hand side of equation (50) converges it follows from (47) that

$$\sum_{j=1}^{\infty} jc_j(t)<\infty \tag{51}$$

for all finite times. (A different type of solution for the case $F=2(i+j)$ has been discussed in [7].)

Remarks

Approach to equilibrium in the discrete version of equation (2) for the pure coagulation case has been discussed in Leyvraz [17] who mentions that the final equilibrium solution consists of one infinite particle while for finite times the average size of a particle remains finite. Mathematically, producing equilibrium solutions which contain singularities (as discussed above) correspond in the coagulation case to the physical production of an infinite particle; in the relevant literature (for example, see van Dongen and Ernst [18]) this resulting phenomenon is called the 'gel' or 'superparticle' solution whose mass is physically comparable to that of the entire system of particles. For comments on the pure fragmentation case see Ziff and McGrady [7]. The inclusion of source terms and the consequence on the coagulation equilibria have been discussed by White [19].

It is hoped that further investigations of singular solutions will be discussed in future work by the authors.

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